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Abstract

We introduce random variables representing technical inefficiency and uncertainty into the problem of maximising the expected utility of profits. We then use the results of Pope and Chavas (1994) to identify cost minimisation problems which are not only consistent with expected utility maximisation but are also independent of risk preferences.

The first-order conditions for our cost minimisation problems are solved for a set of behavioural equations which are functions of deterministic unobserved 'planned outputs' or 'aggregator functions'. In empirical work we can replace these deterministic arguments with observed quantities and, in the process, endow the behavioural equations with a mutually consistent stochastic structure representing technical inefficiency and/or uncertainty. In this way, it is possible to overcome the 'Greene Problem' encountered in the frontier literature.

* Not to be quoted without the author's permission. This research was undertaken while the author was visiting Texas A&M University and the University of Guelph.
1. Introduction

Many econometric models of technically inefficient firm behaviour use as their starting point a set of behavioural equations (eg. cost and cost-share equations) which have been derived from a deterministic producer optimisation problem. Technical inefficiency is then introduced by appending to these equations a carefully chosen set of error terms, some of which may be truncated in order to capture certain stylised economic facts (eg. that technical inefficiency increases costs). Examples include the cost and cost-share equations of Greene (1980) and Ferrier and Lovell (1990). A well-recognised problem with this approach is that the error terms appended to different behavioural equations may not provide internally consistent representations of inefficient behaviour. Greene (1980), for example, assumes that the errors in the cost-share equations are independent of the error in the cost equation, even though incorrect shares are known to increase cost. Ferrier and Lovell (1990) attempt to capture the cost and cost-share effects of technical inefficiency by appending an error term to the cost equation only.

One method of overcoming this so-called 'Greene Problem' is to introduce parameters representing technical inefficiency directly into the optimisation problem, and to carry these parameters into the behavioural equations by way of the first-order optimising conditions. In this way, all behavioural equations are endowed with a mutually consistent representation of inefficiency, albeit a parametric one. Examples of this approach include the cost and cost-share equations of Atkinson and Cornwell (1993, 1994a). A distinctive feature of this approach is that the optimisation problem is still deterministic, so the optimising conditions yield a set of deterministic behavioural equations which must still be 'embedded in a stochastic framework'. Atkinson and Cornwell (1993, 1994a), for example, introduce statistical noise.

A natural extension of this approach is to use 'producer-observable variables' (rather than parameters) to represent technical inefficiency in the optimisation problem. By 'producer-observable variables' we mean variables whose values are known to the producer at the time input decisions are made. To use the terminology of Schmidt (1985-86), technical inefficiencies are 'foreseen'. Examples of this approach include the cost and first-order-optimising conditions of Schmidt and Lovell (1979), and the additive general error models of McElroy (1987) and Kumbhakar (1989). Of course the optimisation problem, the first-order optimising conditions and the behavioural equations are all still deterministic. Most authors, including McElroy (1987), Schmidt and Lovell (1979) and Kumbhakar (1989), 'embed' their behavioural equations in a 'stochastic framework' by assuming (sometimes implicitly) that the 'producer-observable variables' are unobserved by the researcher. In this way, the optimisation problem is used to endow the behavioural equations with a mutually consistent representation of inefficiency, this time a stochastic one.

In this paper we further extend this approach by using 'random variables' (rather than parameters or producer-observable variables) to represent technical inefficiency in the producer optimisation problem. By 'random variables' we mean variables whose values are not known to the producer at the time input decisions are made. Thus, technical inefficiency is 'unforeseen' (Schmidt 1985-86) and the decision-making environment is said to be characterised by 'uncertainty'. The plausibility of unforeseen technical inefficiency has been only briefly discussed in the frontier literature by, for example, Schmidt (1985-6). In contrast, decision-making under uncertainty has been widely discussed in the agricultural economics literature, with important recent contributions by Daughtrey (1982), Pope (1980) and Pope and Chavas (1994). By dealing with technical

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1 Although Schmidt and Lovell (1979) and Kumbhakar (1989) do not make their assumptions regarding (producer) observability of technical inefficiencies explicit, this is implied by the assumption that producers minimise cost. McElroy (1987) does, however, make this assumption explicit, and it is clear that Kumbhakar (1989) relies heavily on this earlier work.

2 Schmidt and Lovell (1979) and Kumbhakar (1989) also introduce statistical noise.
inefficiency in an uncertainty framework we hope to provide an overview of, and perhaps build something of a bridge between, the frontier and uncertainty/agricultural economics literatures.

We take the classical economic theory approach to decision-making under uncertainty by assuming that producers maximise the expected utility of profits. In the past, this assumption has severely limited the empirical convenience and usefulness of duality theory, since cost functions have usually depended on risk preferences. Recently, however, Pope and Chavas (1994) have shown how to remove this dependency, by imposing restrictions on the form of the stochastic revenue function. In this paper we consider two such restrictions, both of which guarantee the existence of a tractable cost function which does not depend on risk preferences, and one of which ensures that random variables representing technical inefficiency are carried into at least some of our behavioural equations. Thus, at least some of our behavioural equations inherit a stochastic structure directly from the producer optimisation problem, and can be estimated without being further 'embedded in a stochastic framework'. When it occurs, this 'stochastic inheritance' has a great deal of theoretical appeal (see McElroy 1987) and at least one important implication for econometric work: full-information estimators (eg. FIML) will be identical for all representations of the technology.

One of the restrictions we consider in this paper may actually prevent this 'stochastic inheritance'. The restriction which does this is one which constrains the variances of all but one random variable to zero. If the variances of all random variables are equal to zero, technical inefficiencies are once again represented by parameters (specifically, the means of the distributions of our random variables) and our behavioural equations collapse to the deterministic behavioural equations found in the frontier literature, such as those derived by Atkinson and Cornwell (1993, 1994a, 1994b). Thus, our paper is not as narrowly focused as it might first appear: our model of producer behaviour under uncertainty is general enough to accommodate much of the frontier literature as a special case.

The outline of the paper is as follows. In Section 2 we provide a brief description of the production technology, which is described in terms of the production possibilities set. Arguably the most important simplifying assumption we make in this Section is that of output separability, which allows us to represent our multiple-output technology in single-output terms. In Section 3 we make specific assumptions concerning uncertainty and inefficiency, a task which is made easier by our single-output representation of the technology. The producer optimisation problem is the subject matter of Section 4, where we specify two restrictions which guarantee that cost minimisation is consistent with maximisation of the expected utility of profits, and which also guarantee that the cost function is not an explicit function of risk preferences. The cost, input-demand and cost-share equations derived under each restriction are presented separately in Sections 5 and 6. In Section 7 we use shadow input prices as a vehicle for introducing allocative inefficiencies into our model, which is then complete. The paper is concluded in Section 8, where we summarise our results and offer suggestions for further research.

2. The Production Technology

We begin by assuming that the firm employs an (Nx1) non-negative real-valued input vector x = (x1, ..., xN)' to produce an (Mx1) non-negative real-valued output vector y = (y1, ..., yM)'. We choose to describe the technology using the production possibilities set T = {(x, y): x can produce y} and we follow Chambers (1988, p.252) by assuming that:

A.1 T is a non-empty, closed, convex set
A.2 If (x, y) ∈ T and x1 ≥ x then (x1, y) ∈ T (free disposability of x)
A.3 If (x, y) ∈ T and y1 ≤ y then (x, y1) ∈ T (free disposability of y)
A.4 (x, 0M) ∈ T
A.5 If y>0, (0_N, y) ∈ T (weak essentiality)
A.6 For every finite x, T is bounded from above

These reasonably straightforward assumptions are a subset\(^1\) of the minimal properties due to Shephard (1970). Assumption A.1 ensures that some feasible input-output combinations exist, that there are no 'holes' in the boundary of T, and that the average of any two technically feasible input-output combinations is also feasible. A.2 and A.3 ensure that extra inputs do not hamper production, A.4 says that producing zero output is always feasible, while A.5 says it is not possible to produce outputs without inputs. A.6 is a regularity condition which guarantees the existence of a transformation function and the existence of well-defined extrema for some of the producer optimisation problems below.

In order to more easily accommodate uncertainty and technical inefficiency within our model, we also assume output separability\(^2\):

A.7 There exists a set T\(^*\) and a finite, non-negative, real-valued index q(y) such that (x, q(y)) ∈ T\(^*\)

\[ \Rightarrow (x, y) \in T \]

This assumption is popular in the applied agricultural economics literature (eg. Ball and Chambers 1982) and means we can represent our multiple-output technology in single-output terms (see Chambers 1988 p.285):

\[
\begin{align*}
(1) \quad f(x) &= \max \{ q(y) : (x, q(y)) \in T^* \} \\
&= \max \{ q : (x, q) \in T^* \}
\end{align*}
\]

where q = q(y). Assumptions A.1 to A.6 imply that the input requirement set V(q) = {x : f(x)≥q} is closed and non-empty for all q>0, and that f(x) satisfies monotonicity, quasi-concavity and weak essentiality (Chambers 1988 p.258-9). Moreover, since q(y) is finite, non-negative and real-valued, f(x) is also finite, non-negative and real-valued for all finite and non-negative x.

Finally, to permit the use of differential calculus, and to allow us to establish consistency between cost minimisation and maximisation of the expected utility of profits, we assume:

A.8 f(x) is everywhere continuous and everywhere twice-continuously differentiable.
A.9 f(x) is nonrandom and does not depend on risk preferences

3. Technical Inefficiency and Uncertainty

An input-output combination (x̂, ̂q) is technically efficient if and only if there is no (x, q) ∈ T\(^*\) such that x ≤ x̂ and q ≥ ̂q; that is, if there is no way to produce more output with the same inputs or to produce the same output with fewer inputs (Varian 1992, p.4). Clearly, for a technically efficient input-output combination (x̂, ̂q) we can write:

\[ x \leq x^* \quad \text{and} \quad q \geq q^* \]

We do not assume, for example, that if x>0 there exists λ such that (λx, y) ∈ T for all finite y≥0.

Another assumption which would have allowed us to accommodate uncertainty and inefficiency is the assumption of input non-jointness: for every (x, y) ∈ T there exist vectors x\(^m\)≥0 such that y\(_m\)≤f\(_m\)(x\(^m\)) and Σx\(^m\) ≤ x, where f\(_m\)(.) is a production function satisfying weak monotonicity, weak essentiality, nonemptiness, and closedness of the input requirement set (see Chambers 1988, p.287). The implications of input non-jointness for the stochastic structure of transformation, input-demand, cost-share and cost functions are addressed in another paper.
In the remainder of this paper we shall use overbar notation to denote input-output combinations which are technically efficient. Moreover, we shall make explicit the relationships between technically efficient and inefficient input-output combinations by assuming that for all \((x, q) \in T^*\):

\[
\begin{align*}
A.10 & \quad x_n = \bar{x}_n u_n \\
A.11 & \quad q = \bar{q} \theta \\
\end{align*}
\]

where \(\theta\) and \(u_n\) \((n = 1, \ldots, N)\) are finite and real-valued. Clearly an input-output combination \((x, q) \in T^*\) is technically efficient if \(\theta = u_n = 1\) for all \(n\).

Various parameterisations of assumptions A.10 and A.11 can be found in the frontier literature, where equivalents of \(\theta\) and \(u_n\) \((n = 1, \ldots, N)\) are (usually implicitly) assumed to be 'foreseen'. Assumption A.10 has been maintained by, for example, Kumbhakar (1988) who uses an equivalent of \(u_n\) to reflect 'input-specific technical inefficiency' (relative to some fixed factor). Atkinson and Cornwell (1993, 1994a, 1994b) also maintain A.10, but impose the restriction that \(u_n = u\) for all \(n\) (i.e. technical inefficiency is not input-specific). Assumption A.11 has been maintained by Schmidt and Lovell (1979) and Atkinson and Cornwell (1993, 1994a, 1994b), with their equivalents of \(\theta\) usually being referred to as 'output technical inefficiency'.

Slightly different interpretations have emerged in the uncertainty literature, where the random variables \(\theta\) and \(u_n\) \((n = 1, \ldots, N)\) are assumed to be 'unforeseen'. Walters (1960), for example, maintains A.10 but uses his equivalent of \(u_n\) to represent 'random variations in flows of factor services'. O'Donnell and Woodland (1995) maintain an unrestricted\(^1\) version of A.11 and use \(\theta\) to represent 'the random effects of industrial disputes, weather and disease'. Some of these interpretations are consistent with those from the frontier literature (e.g. reductions in flows of factor services can be viewed as a form of input technical inefficiency) but some are not (e.g. variations in weather would not generally be regarded as a source of output technical inefficiency). In most sections of this paper we shall attempt to accommodate a multiplicity of plausible interpretations by referring to \(\theta\) and \(u_n\) \((n = 1, \ldots, N)\) as an 'output scaling factor' and 'input scaling factors' respectively.

Together with equation (2), assumptions A.10 and A.11 imply that for all \((x, q) \in T^*\) the production function can be written:

\[
q = f(x_1/v_1, \ldots, x_N/v_N) \theta
\]

\[
= m(n(x_1/v_1, \ldots, x_N/v_N)) \theta \quad \text{if } f(\cdot) \text{ is homothetic.}
\]

\[
= m(n(x)/v) \theta \quad \text{if } u_n = u \forall n \text{ and } f(\cdot) \text{ is homothetic.}
\]

\[
= f(x) \theta/v^k \quad \text{if } u_n = u \forall n \text{ and } f(\cdot) \text{ is homogeneous of degree } k
\]

where \(m(\cdot)\) is a monotonic function and \(n(\cdot)\) is a function which is homogenous of degree one\(^2\). Thus, the production function for the (inefficient) firm (in the presence of uncertainty) can be viewed as a standard (i.e. deterministic and technically efficient) production function evaluated at inputs \((x_1/v_1, \ldots, x_N/v_N)\) and post-multiplied by \(\theta\).

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1. \(\theta\) is not restricted to lie on the unit interval.

2. There is no loss of generality in our restriction that \(n(\cdot)\) be homogenous of degree one: a monotonic transform of an homogenous function of degree \(k>0\) can always be expressed as a monotonic transform of an homogenous function of degree one (Shephard 1970 p.31).
The empirical implications of equation (3) are twofold. First, both the output scaling factor ($\theta$) and the input scaling factors ($\nu_n$) should appear in the production function, and in a form which is determined by the homotheticity and homogeneity properties of the technology. Second, if $\nu_n = \nu \forall n$ and $f(.)$ is homogeneous of degree $k$ then the output and input scaling factors will be underidentified (they may also be underidentified if $\nu_n = \nu \forall n$ and $f(.)$ is homothetic). A large number of frontier equations with an underlying functional structure given by (3) have been estimated under the assumptions that $\nu_n = \nu \forall n$ and $f(.)$ is homogeneous. Examples include the production frontiers of Aigner, Lovell and Schmidt (1977), Pitt and Lee (1981), Schmidt and Sickles (1984), Battese and Coelli (1988), Kumbhakar (1990) and Dawson, Lingard and Woodford (1991).

4. Optimising Behaviour

Although the theoretical economics literature contains a number of alternative descriptive models of choice under uncertainty (eg. Quiggin 1982) empirical researchers usually maintain the assumption of maximisation of expected utility, partly because it provides for easy parameterisation of preferences. Recent examples from the applied agricultural economics literature include the studies by Chavas and Holt (1990) and Duffy, Shalishali and Kinnucan (1994). This paper continues in the tradition established in the empirical literature by assuming that producers maximise the expected utility of profits:

$$A.12 \quad U = \max_{x, q} \mathbb{E}\{U(w_0 + pq - w'x; (x, q) \in T^*)\} = \max_{x \geq 0} \mathbb{E}\{U(w_0 + pf(x_1/\nu_1, x_2/\nu_2, ..., x_N/\nu_N)\theta - w'x)\}$$

where $U(.)$ is a von Neuman-Morgenstern utility function representing risk preferences, $w_0$ is initial wealth, $p$ is a nonnegative index of random output prices\(^1\), and $w$ is an $(N \times 1)$ real-valued vector of non-negative non-random input prices. Prices and initial wealth are regarded as exogenous, and we follow Pope and Chavas (1994) by assuming that:

$$A.13 \quad U(W) \text{ is differentiable and increasing in } W, \text{ and}$$

$$A.14 \quad \mathbb{E}\{U(W)\} \text{ is concave in } W.$$ 

Result 3 of Pope and Chavas (1994) can be used to show that the expected utility maximisation problem A.12 implies cost minimisation if either or both of the following restrictions hold:

- **R.1** $\nu = (\nu_1, ..., \nu_N)'$ is nonstochastic $\Leftrightarrow \mathbb{E}\{\nu'\nu\} = 0$, and
- **R.2** $f(x_1/\nu_1, ..., x_N/\nu_N)$ can be written $f(x_1/\nu_1, ..., x_N/\nu_N) = h(g(x), \nu)$, where $g(x)$ is non-negative, finite, real-valued and differentiable.

These restrictions not only ensure the existence of a deterministic cost minimisation problem, they imply restrictions on its structure. Specifically, if R.1 is satisfied the cost minimisation problem is independent of risk preferences and takes the form:

$$A.15 \quad (4) \quad c(w_1\nu_1, ..., w_N\nu_N, q) = \min_{x \geq 0} \{w'x; q = f(x_1/\nu_1, ..., x_N/\nu_N)\}$$

which implies the expected utility maximisation problem A.12 can be rewritten:

$$A.16 \quad (5) \quad U = \max_{q \geq 0} \mathbb{E}\{U(w_0 + pq\theta - c(w_1\nu_1, ..., w_N\nu_N, q)\)$$

---

\(^1\) In this draft paper we have not considered the conditions required for the existence of the price aggregate $p$. 

If R.2 is satisfied the cost minimisation problem is again independent of risk preferences, and is defined over the nonstochastic aggregator function $\tilde{g} = g(x)$:

$$\tilde{c}(w, \tilde{g}) = \min_{x \geq 0} \{ w^\prime x : \tilde{g} = g(x) \}$$

which implies the expected utility maximisation problem A.12 can be written:

$$U = \max_{\tilde{g} \geq 0} E\{U(w_0 + \phi_0(\tilde{g}, v) - \tilde{c}(w, \tilde{g}))\}$$

Clearly, the Pope and Chavas result implies that the producer optimisation problem can be broken into two parts: first, the producer chooses a so-called 'planned output' ($\tilde{q}$ and $\tilde{g}$ in the case of R.1 and R.2 respectively) to maximise the expected utility of profits (as described by (5) and (7)); and second, the producer chooses inputs to minimise the cost of producing planned output (as described by (4) and (6)). This separation of the expected utility maximisation problem from the deterministic cost minimisation problem has two important implications for empirical work: first, we can estimate a cost function and/or derivatives of it without having to make assumptions about the functional form of $U(.)$ (ie. without having to make assumptions about risk preferences); and, second, it allows us to treat planned and realised outputs as exogenous, which means, for example, that we may be able to estimate some of our cost, input-demand and cost-share equations in a Seemingly Unrelated Regressions (SUR) framework, rather than in a simultaneous equations framework.

5. Cost, Input-Demand And Cost-Share Equations Under R.1: Nonstochastic $v$

If R.1 is satisfied the cost function takes the form $c(w_1, v_1, ..., w_N v_N, \tilde{q})$ and specifies the minimum cost of producing planned output level $\tilde{q}$ at prices $w$ and input scaling factors $v \geq 1_N$. The increase in cost due to input uncertainty/inefficiency is $\Delta c = c(w_1, v_1, ..., w_N v_N, \tilde{q}) - c(w, \tilde{q})$, and Euler's Theorem can be used to show that this reaches a minimum of zero when $v = 1_N$ (details are provided in a Technical Appendix available on request). Under our earlier assumptions on the technology it is also possible to show that the cost function has a standard set of properties (Chambers 1988 p.51-52): nonnegative for all $w, \tilde{q} > 0$; nondecreasing in $w, v$ and $\tilde{q}$; concave and continuous in $w$; positively linearly homogenous in $w$ and $v$; and no fixed costs (ie. $c(w_1, v_1, ..., w_N v_N, 0) = 0$). Thus, $c(w_1, v_1, ..., w_N v_N, \tilde{q})$ may be viewed as a standard (ie. deterministic and technically efficient) cost function which is simply evaluated at planned output level $\tilde{q}$ and shadow prices $(w_1, v_1, ..., w_N v_N)$. Of course planned output coincides with realised output when $\theta = 1$, and shadow prices equal observed prices when $v = 1_N$.

Differentiability of $f(x)$ means the first order conditions for an interior solution (ie. $x > 0$) to the cost minimisation problem (4) can be obtained as:

$$w_n = \lambda f_n(x_1/\tilde{v}_1, ..., x_N/\tilde{v}_N)/\tilde{v}_n = 0$$

$$\tilde{q} = f(x_1/\tilde{v}_1, ..., x_N/\tilde{v}_N)$$

where $f_n(x)$ denotes the partial derivative of $f(x)$ with respect to $x_n$. Moreover, closedness and convexity of $V(\tilde{q})$ implies that a unique solution exists (see Varian 1992 p.53-4), with conditional factor demand equations given by ($n = 1, ..., N$):

$$x_n = x_n(w_1 v_1, ..., w_N v_N, \tilde{q}) u_n$$

$$= x_n(w_1 v_1, ..., w_N v_N, 1) m^{-1}(\tilde{q}) u_n$$

if $f(.)$ is homothetic.
These equations have standard properties (Chambers 1988, p.64): homogeneity of degree zero in \( w \) and \( v \), and the substitution matrix, with elements \( \frac{\partial X_n(w_1v_1, \ldots, w_Nv_N, \bar{q})}{\partial w_k} = \frac{\partial X_n(w_1v_1, \ldots, w_Nv_N, \bar{q})}{\partial v_n} \), is negative semi-definite. Thus, the \( n \)th input demand function for the (inefficient) firm (in the presence of uncertainty) can be viewed as a standard input demand function evaluated at planned output level \( \bar{q} \) and shadow prices \( (w_1v_1, \ldots, w_Nv_N) \), and post-multiplied by the nonstochastic scalar \( u_n \).

In empirical work we replace the unobserved planned output \( \bar{q} \) with \( q/\theta \) (=\( \tilde{q} \)), thereby introducing the one-sided \( \theta \) into the empirical factor demand equations. Thus, like the production function, both the output scaling factor \( (\theta) \) and the input scaling factors \( (u_n) \) should appear in the input-demand functions, and in a manner which is determined by the homotheticity and homogeneity properties of the technology. For example, if \( u_n = v \) \( \forall \ n \) and \( f(.) \) is homogeneous of degree \( k \), then the output and input scaling factors are underidentified, and should appear in the unit input demand functions multiplicatively.

A further implication of equation (9) is that if \( u_n = v \) \( \forall \ n \) (ie. the input scaling factors are input-specific) then \( \frac{\partial x_n(w_1v_1, \ldots, w_Nv_N, \bar{q})}{\partial w_k} = \frac{\partial x_n(w_1v_1, \ldots, w_Nv_N, \bar{q})}{\partial v_k} \) for \( k=\neq n \), and input uncertainty/inefficiency will cause input demands to increase or decrease in line with the complementarity/substitutability relationships between the inputs. Thus, for example, the inefficient use of labour will reduce the demand for labour-substitutes, such as capital.

Finally, the \( n \) input demand functions can be used to recover the cost function:

\[
(11) \quad c = w'x = \sum_{n=1}^{N} (w_nv_n)x_n(w_1v_1, \ldots, w_Nv_N, \bar{q})
\]

\[
= c(w_1v_1, \ldots, w_Nv_N, \bar{q})
\]

\[
= c(w_1v_1, \ldots, w_Nv_N, 1)q^{-1}(\bar{q}) \quad \text{if } f(.) \text{ is homothetic}
\]

\[
= c(w_1v_1, \ldots, w_Nv_N, 1)q^{1/k} \quad \text{if } f(.) \text{ is homogenous of degree } k
\]

\[
= c(w, \bar{q}v) \quad \text{if } u_n = v \ \forall \ n
\]

\[
= c(w_1v_1, \ldots, w_Nv_N, 1)m^{-1}(\bar{q})v \quad \text{if } u_n = v \ \forall \ n \text{ and } f(.) \text{ is homothetic}
\]

\[
= c(w, 1)q^{1/k}v \quad \text{if } u_n = v \ \forall \ n \text{ and } f(.) \text{ is homogenous of degree } k
\]

and the input cost-share equations:
Not surprisingly, the cost-share equations have standard properties: nonnegative for all \( w, \hat{q} > 0 \), and homogeneous of degree zero in \( w \) and \( v \). Thus, both the cost and cost-share equations can be viewed as standard equations evaluated at planned output level \( \hat{q} \) and shadow prices \( (w_1v_1, \ldots, w_Nv_N) \).

Equations (11) and (12) reveal that replacing the unobserved \( \hat{q} \) with \( q/\theta = \hat{q} \) may have different implications for the stochastic structures of the cost and cost-share equations. Specifically, this substitution always introduces \( \theta \) into the cost function, but it fails to introduce \( \theta \) into any cost-share equations derived from an homogenous or homothetic production function. Thus, since \( \theta \) is the only possible source of stochastic variation in the cost-share equations (\( v \) is nonstochastic by assumption), the share equations derived from an homogenous or homothetic production function are inestimable without appending some form of error term. Of course, if \( \theta \) is nonstochastic then, irrespective of the homogeneity properties of the technology, all our behavioural equations are deterministic and must be 'embedded in a stochastic framework'. In fact, our cost and cost-share equations collapse to the parametric frontiers of Atkinson and Cornwell (1993, 1994a) who, it will be recalled, introduce random variables representing statistical noise.

6. Cost, Input-Demand And Cost-Share Equations Under R.2: \( f(x_1/v_1, \ldots, x_N/v_N) = h(g(x), v) \)

If R.2 is satisfied the cost function takes the form \( \hat{c}(w, \hat{g}) \) and specifies the minimum cost of producing planned output level \( \hat{g} \) at prices \( w \). Note that \( \hat{c}(.) \) and \( c(.) \) are different functions, and that \( \hat{c}(.) \) is not defined over the stochastic \( v \). Although \( \hat{c}(.) \) is not defined over \( v \), it is possible to show that the increase in cost due to input uncertainty/inefficiency, \( \Delta c = \hat{c}(w, \hat{g}) - c(w, \hat{q}) \), reaches a minimum of zero when \( v = t_N \) (again, details are available in a Technical Appendix available on request). It is also possible to show that the input requirement set \( V(\hat{g}) = \{x: g(x) \geq \hat{g}\} \) is closed and non-empty for all \( \hat{g} > 0 \), and that \( g(x) \) satisfies monotonicity and weak essentiality. Thus, the cost function \( \hat{c}(w, \hat{g}) \) will have the standard properties (Chambers 1988 p.51-2; Pope and Chavas 1994 p.199): nonnegative for all \( w, \hat{g} > 0 \); nondecreasing in \( w \); concave and continuous in \( w \); positively linearly homogenous in input prices; nondecreasing in \( \hat{g} \); and no fixed costs \( \hat{c}(w, 0) = 0 \). Thus, \( \hat{c}(w, \hat{g}) \) can be also regarded as a standard cost function, on this occasion evaluated at the 'planned output' level \( \hat{g} \).

Differentiability of \( f(x) \) implies differentiability of \( g(x) \), so the first order conditions for an interior solution to the cost minimisation problem (6) can be obtained as:

\[
\begin{align*}
    w_n - \lambda g_n(x) &= 0 & n = 1, \ldots, N, \\
    \hat{g} &= g(x)
\end{align*}
\]
where $g(X)$ denotes the partial derivative of $g(x)$ with respect to $X$.

Again, closedness and convexity of the input requirement set implies these first order conditions can be solved to yield the conditional factor demand equations ($n = 1 \ldots N$):

\[(14) \quad x_n = \frac{\partial W}{\partial x_n}/\frac{\partial W}{\partial w} = X_n W_1^{m-1}(\beta \cdot v)\]

if $X_n = X$ and $f(.)$ is homothetic.

if $f(.) = 1$ and $f(.)$ is homogeneous of degree $k$.

Again, these equations have standard properties: homogeneity of degree zero in $w$; and the substitution matrix, with elements $\frac{\partial x_n(W, \tilde{w})}{\partial w}$, is negative semi-definite. Thus, the $n\textsuperscript{th}$ input demand function for the (inefficient) firm can be viewed as a standard input demand function evaluated at $\tilde{w}$ rather than $q$. Of course, in empirical work we replace $\tilde{w}$ with $h^{-1}(q, \tilde{w})$, so the empirical factor demand equations once more become functions of $\tilde{w}$ and $\beta$. Not surprisingly, the conclusions we reached in Section 5 concerning homogeneity, homotheticity and the specificity of the input scaling factors are still valid.

Finally, the factor demand equations can be used to recover the cost function:

\[(15) \quad c = w'x = \sum_{n=1}^{N} w_n x_n = C'(w, \tilde{w}) \]

if $X_n = X$ and $f(.)$ is homothetic.

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The share equations are nonnegative for all \( w, \hat{g} > 0 \) and are homogenous of degree zero in prices. Thus, once again, both the cost and cost-share equations can be viewed as standard equations evaluated at the planned output level \( \hat{g} \) (which, in empirical work, we replace with \( h^{-1}(q, \nu)\delta^{-1} \)).

It is worth noting from equations (14), (15) and (16) that if \( v_n = v \forall n \) and \( f(x) \) is either homothetic or homogenous of degree \( k \), then the cost, input-demand and cost-share equations obtained under R.2 can be expressed as functions of the cost, input-demand and cost-share equations obtained under R.1. This is not surprising since, if \( v_n = v \forall n \) and \( f(x) \) is either homothetic or homogenous of degree \( k \), the input and output scaling factors are underidentified. Furthermore, because output scaling factors play no role in establishing consistency between cost minimisation and maximisation of the expected utility of profits, it is immaterial whether R.1, R.2, both or neither are satisfied.

7. Allocative Inefficiency

A cost minimising firm is allocatively inefficient if and only if it operates off its least cost expansion path, i.e. if it produces output \( q \) (or \( \hat{q} \) or \( \hat{g} \)) using input combinations which do not minimise the cost of producing \( q \) (or \( \hat{q} \) or \( \hat{g} \)). Thus, allocative inefficiency implies that the first order conditions are not met exactly.

A convenient way of motivating allocative inefficiencies is to assume that producers maximise the expected utility of profit subject to \( S \) new constraints \( r_s(w, x) = 0 \) (\( s = 1, ..., S \)) (in addition to the usual technology constraint \( (x, q) \in T^* \)). These new constraints reflect the pursuit of non-expected-utility-maximising behaviour: they may, for example, reflect government regulations, or dynamic considerations in the choice of some inputs. Importantly, under R.1 and R.2, producers still minimise costs, with the first \( N \) first-order-conditions for an interior solution now reflecting the new constraints (\( n = 1, ..., N \)):

\[
(17) \quad w_n^* - \lambda f_n(x_1/v_1, ..., x_N/v_N)/v_n = 0
\]

in the case of R.1, and

\[
(18) \quad w_n^* - \lambda g_n(x) = 0
\]

in the case of R.2, where

\[
(19) \quad w_n = w_n - \sum_{s=1}^{S} \lambda_s \partial r_s(w, x)/\partial x_n
\]

By analogy with equations (8) and (13), it is clear that allocative inefficiencies can be introduced into the theoretical input-demand equations of Sections 5 and 6 by simply replacing the vector of observed prices \( w \) with the vector of shadow prices \( w^* = (w_1^*, ..., w_N^*)' \). In empirical work it is often useful to parameterise optimisation (allocation) errors as

\[
(20) \quad \kappa_n = 1 - \sum_{s=1}^{S} \lambda_s \partial r_s(w, x)/\partial x_n
\]

so that (19) can be rewritten:

\[
(21) \quad w_n^* = w_n \kappa_n \quad \kappa_n \geq 0
\]
Notice that the condition $\kappa_n \geq 0$ ensures that the shadow prices $w^*$ will be non-negative, and that a firm will be allocatively efficient if and only if $\kappa_n = \kappa \forall n$. Equation (21) has been used to model the relationship between observed and shadow prices by, for example, Toda (1976), Atkinson and Halvorsen (1984, 1990), Lovell and Sickles (1983) and Kumbhakar (1992).

Equation (21) implies that our cost functions can be recovered as:

\[
(20) \quad c = c(w_1^*, \ldots, w_N^*, \tilde{g}) \sum_{n=1}^{N} s_n(w_1^*, \ldots, w_N^*, \tilde{g}) \kappa_n^{-1}
\]

in the case of R.1 and

\[
(21) \quad c = \bar{c}(w^*, \tilde{g}) \frac{N}{n=1} \bar{s}_n(w^*, \tilde{g}) \kappa_n^{-1}
\]

in the case of R.2. Moreover, the share equations can be recovered as:

\[
(22) \quad s_n = \frac{s_n(w_1^*, \ldots, w_N^*, \tilde{g}) \kappa_n^{-1}}{\sum_{j=1}^{N} s_j(w_1^*, \ldots, w_N^*, \tilde{g}) \kappa_j^{-1}}
\]

in the case of R.1 and

\[
(23) \quad s_n = \frac{\bar{s}_n(w^*, \tilde{g}) \kappa_n^{-1}}{\sum_{j=1}^{N} \bar{s}_j(w^*, \tilde{g}) \kappa_j^{-1}}
\]

in the case of R.2. Thus, the cost and cost-share equations can be regarded as standard cost and cost-share equations evaluated at shadow prices and different outputs, and multiplied or divided by a share weighted average of the inverse allocative inefficiencies.

Two final observations are in order. First, the increase in cost due to allocative inefficiencies is non-negative and reaches a minimum of zero when $\kappa_n = \kappa \forall n$. Second, equation (21) collapses to equation (20), and (23) collapses to (22), if $\tilde{v}_n = \tilde{v}_n$ and $f(x)$ is homogenous of degree $k$. In this case, allocative inefficiencies and the input and output scaling factors are all underidentified.

8. Conclusion

Stochastic specification is an important part of applied econometric work. Unfortunately, as McElroy (1987 p.738) succinctly points out, "most of this work proceeds as if the static neoclassical model of a ... firm has no implications for the specification of the stochastic parts of the implied [behavioural] equations (cost function, input demand, and share functions), and vice versa". In contrast, this paper introduces random variables into the problem of maximising the expected utility of profits, and carries these random variables into the behavioural equations by way of the first-order optimising conditions. Thus, most of our cost, cost-share and input-demand functions inherit a mutually consistent stochastic structure directly from our static model of firm behaviour. In this way, we overcome the 'Greene Problem'.

The behavioural equations implied by our model of firm behaviour can be viewed as standard (ie. technically efficient and deterministic) behavioural equations evaluated at shadow prices and planned outputs, and sometimes
multiplied by (functions of) random variables. Thus, in empirical work we can choose a functional form for our cost, cost-share and input-demand systems from the full menu of functional forms commonly used in applied production economics. This includes exact (eg. Cobb-Douglas and Constant Elasticity of Substitution) as well as flexible (eg. Translog, Generalised Leontief and Generalised Quadratic) functional forms. Importantly, our choice is not limited to self-dual functional forms (eg. Cobb-Douglas) as Atkinson and Cornwell (1994a p. 233) and Bauer (1990 p.44) have suggested. However, our choice may be limited by the fact that our behavioural equations inherit a stochastic structure from the optimisation problem, making it more difficult to craft a combination of functional form and distributional assumptions which will make our (systems of) equations empirically tractable. The problem is likely to be greater for multiple-equation rather than single-equation models (because we need to craft multiple rather than single combinations of assumptions), for cost plus cost-share systems rather than input-demand systems (because the structure of the cost function will differ from the structure of the cost-share functions, whereas the structure of each input demand function will be identical), and for models in which all scaling factors and allocative inefficiencies, rather than just some, are assumed to be random (because some functional forms will give rise to nonlinearities, and a function with a nonlinear deterministic component is often easier to estimate than a model with a nonlinear error term having moments that are difficult or impossible to derive). The problem may also be greater when using flexible functional forms: in such cases we can only guarantee that our behavioural equations are consistent with maximisation of the expected utility of profits by assuming the input scaling factors are non-random or, if the production function is homogenous, by assuming they are not input-specific (ie. $\sigma_n = \sigma$ $\forall$ $n$).

A cornerstone of our analysis is the assumption that the inputs and outputs of a technically efficient firm can be represented by the very general production function $\mathbf{q} = f(\mathbf{x})$. The generality of this assumption is then diminished by our assumptions concerning inefficiency and uncertainty, leading us to specify a relationship between observed inputs and outputs of the form $\mathbf{q} = f(\mathbf{x}_1/\mathbf{v}_1, ..., \mathbf{x}_N/\mathbf{v}_N)\theta$. It is, however, quite easy to motivate and accommodate other inefficiency/uncertainty assumptions within our methodological framework, leading to a wide range of other input-output relationships, including a variant of the production functions used by Just and Pope (1978) and Griffiths and Anderson (1982): $\mathbf{q} = f(\mathbf{x}_1/\mathbf{v}_1, ..., \mathbf{x}_N/\mathbf{v}_N) + \mathbf{h}(\mathbf{x}_1/\mathbf{v}_1, ..., \mathbf{x}_N/\mathbf{v}_N)\theta$. These alternative uncertainty/inefficiency assumptions provide clear opportunities for further theoretical and empirical research.
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